From these equations we find the values of A, B, C, D, E; whence,

$$U_n = -20 + 36n - \frac{115n^2}{6} + \frac{9n^3}{2} - \frac{n^4}{3}.$$

From this we get the 6th term to be 46.

Thus, the series is

$$1 + 6 + 10 + 20 + 35 + 46 + \cdots.$$

Using the method of differences we get

$$\begin{array}{cccccc} 5 & 4 & 10 & 15 & 11 & \cdots \\ & -1 & 6 & 5 & -4 & \cdot \cdots \\ & & 7 & -1 & -9 & \cdot \cdot \cdots. \\ & & & -8 & -8 & \cdot \cdot \cdot \cdots \\ & & & & 0 & \cdot \cdot \cdot \cdot \cdots \end{array}$$

Then

$$S_n = n + \frac{5n(n-1)}{\lfloor 2} - \frac{n(n-1)(n-2)}{\lfloor 3} + \frac{7n(n-1)(n-2)(n-3)}{\lfloor 4} - \frac{8n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}.$$

$$= -\frac{n}{120}(8n^4 - 115n^3 + 510n^2 - 1145n + 622).$$

(c) In the series $1 + 5 + 15 + 35 + 70 + \cdots$

$$U_n = A + Bn + Cn^2 + Dn^3 + En^4 + \cdots.$$

$$\left\{\begin{array}{l} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 5, \\ A + 3B + 9C + 27D + 81E = 15, \\ A + 4B + 16C + 64D + 256E = 35, \\ A + 5B + 25C + 125D + 625E = 70, \end{array}\right\} \begin{array}{l} A = 0, \\ B = 1/4, \\ C = 11/24, \\ D = 1/4, \\ E = 1/24. \end{array}$$

Hence,

$$U_n = \frac{n}{4} + \frac{11n^2}{24} + \frac{n^3}{4} + \frac{n^4}{24}.$$

Then

$$24S_n = 6\Sigma n + 11\Sigma n^2 + 6\Sigma n^3 + \Sigma n^4,$$

$$24S_n = 3n(n+1) + \frac{11n(n+1)(2n+1)}{6} + \frac{3n^2(n+1)^2}{2} + \frac{n(n+1)(6n^3+9n^2+n-1)}{30}.$$

$$S_n = \frac{n}{120}(n^4 + 10n^3 + 35n^2 + 50n + 24) = \frac{n}{120}(n+1)(n+2)(n+3)(n+4).$$

Also solved by O. S. Adams, H. H. Conwell, Paul Capron, and William Tier.

GEOMETRY.

496. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

Solution by William Hoover, Columbus, Ohio.

One obvious solution is given when one of the tangent planes is fixed in position, for then this plane is the locus of lines common to it and a second tangent plane orthogonal to the first.

It is not troublesome to show that the equation of a first plane embracing a line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \tag{1}$$

$$\frac{(x-a)\lambda_1}{l} + \frac{(y-b)\mu_1}{m} + \frac{(z-c)\nu_1}{n} = 0, \tag{2}$$

with the condition

$$\lambda_1 + \mu_1 + \nu_1 = 0, \tag{3}$$

and that of a second plane embracing (1) is

$$\frac{(x-a)\lambda_2}{l} + \frac{(y-b)\mu_2}{m} + \frac{(z-c)\nu_2}{n} = 0, \tag{4}$$

with the condition

$$\lambda_2 + \mu_2 + \nu_2 = 0. \tag{5}$$

If (2) and (4) are orthogonal,

$$\frac{\lambda_1}{l}\cdot\frac{\lambda_2}{l} + \frac{\mu_1}{m}\cdot\frac{\mu_2}{m} + \frac{\nu_1}{n}\cdot\frac{\nu_2}{n} = 0, \tag{6}$$

and if (2) and (4) touch

$$a_1x^2 + b_1y^2 + c_1z^2 = 1, \tag{7}$$

$$\left(\frac{\lambda_1}{l}\right)^2\Big/a_1 + \left(\frac{\mu_1}{m}\right)^2\Big/b_1 + \left(\frac{\nu_1}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_1}{l} + \frac{b\mu_1}{m} + \frac{c\nu_1}{n}\right)^2, \tag{8}$$

and

$$\left(\frac{\lambda_2}{l}\right)^2\Big/a_1 + \left(\frac{\mu_2}{m}\right)^2\Big/b_1 + \left(\frac{\nu_2}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_2}{l} + \frac{b\mu_2}{m} + \frac{c\nu_2}{n}\right)^2. \tag{9}$$

Now (3), (5), (6), (8), (9) is a system of *five* equations for the determination of the *six* ratios λ_1/ν_1, μ_1/ν_1; λ_2/ν_2, μ_2/ν_2; and l/n, m/n, giving an indeterminate solution. The values of l/n, m/n are the only ones needed in (1).

There seems to be a missing condition in the statement of the problem.

If the direction of the line (1) were constant, or $l : m : n$, constant, the *locus* of the line (1) would be a right circular cylinder, or, in other words, the locus of the line of intersection of constant direction of pairs of orthogonal tangent planes to a central conicoid is a right circular cylinder.

Note.—The single condition on the line restricts it to be a member of a line complex whose order apparently may be as high as 8. It would be desirable to determine this explicitly. EDITORS.

497. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two spheres on any line passing through any point common to the two spheres.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let the plane of the common circle of the spheres be chosen for yz-plane, and the line of centers for x-axis. Let the radius of the common circle be k, and let $(a, 0, 0)$ and $(b, 0, 0)$ be the centers of the two spheres. Then the equation of one sphere is

$$(x-a)^2 + y^2 + z^2 = a^2 + k^2,$$

or

$$x^2 + y^2 + z^2 - 2ax = k^2; \tag{1}$$

and, similarly, the equation of the other is

$$x^2 + y^2 + z^2 - 2bx = k^2. \tag{2}$$

Let l, m, n be the direction cosines of an arbitrary line through the point M $(0, 0, k)$. Then the equation of the line may be written

$$\frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} = r, \tag{3}$$

where r is the length of the segment joining $(0, 0, k)$ and (x, y, z).

To find the length of the segment MP cut from this line by the sphere (1), we substitute $x = lr$, $y = mr$, $z = nr + k$ in equation (1) and solve for that value of r which is not zero. We thus find at once

$$r = MP = 2al - 2kn.$$

Substituting the same values in (2), we find likewise for the segment MQ intercepted by the